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Wilson loops in 2D noncommutative euclidean gauge theory
1. Perturbative expansion

Jan Ambjørn,[ac] **Andrey Dubin**[b] **and Yuri Makeenko**[ab]

[a] *The Niels Bohr Institute*
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
[b] *Institute of Theoretical and Experimental Physics*
B. Cheremushkinskaya 25, 117259 Moscow, Russia
[c] *Institute for Theoretical Physics, Utrecht University*
Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands
E-mail: ambjorn@nbi.dk, dubin@heron.itep.ru, makeenko@itep.ru

Abstract: We calculate quantum averages of Wilson loops (holonomies) in gauge theories on the euclidean noncommutative plane, using a path-integral representation of the star-product. We show how the perturbative expansion emerges from a concise general formula and demonstrate its anomalous behavior at large parameter of noncommutativity for the simplest nonplanar diagram of genus 1. We discuss various UV/IR regularizations of the two-dimensional noncommutative gauge theory in the axial gauge and, using the noncommutative loop equation, construct a consistent regularization.

Keywords: Field Theories in Lower Dimensions, 1/N Expansion, Non-Commutative Geometry.

Contents

1. Introduction

Noncommutative field theories attracted a lot of attention when they appeared in the context of M(atrix) Theory [1] and certain string models [2] (see [3] for a review and references therein). However, the definition and study of these theories predates [1, 2] and is an interesting subject in its own right (for a recent review see [4]). In certain sense these theories provide us with a minimal quasi-local extension of ordinary local field theories, which remains tractable in a number of ways.

In short, given a commutative field theory defined in euclidean space $\mathbb{R}^D$ by the action

$$S = \int \mathrm{d}^D x \, \mathcal{L}(\phi(x)) \,, \tag{1.1}$$

the corresponding noncommutative theory is implemented by modifying products of the fields $\phi(\mathbf{x})$ to so-called star-products, introduced according to the rule

$$(f_1 * f_2)(\boldsymbol{x}) \equiv \exp\left(-\frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial^y_\mu\partial^z_\nu\right) f_1(\boldsymbol{y})\, f_2(\boldsymbol{z})\Big|_{y=z=x} . \tag{1.2}$$

In (1.2) $\theta_{\mu\nu}$, the so-called parameter of noncommutativity, which enters the commutation relation

$$[\hat{x}_\mu, \hat{x}_\nu] = -\mathrm{i}\,\theta_{\mu\nu}\hat{1} \tag{1.3}$$

is real and antisymmetric.

In particular, the action of the standard Yang-Mills theory is changed to

$$S = \frac{1}{4g^2}\int \mathrm{d}^D x\ \mathrm{tr}\left(\mathcal{F}^2_{\mu\nu}(\boldsymbol{x})\right), \tag{1.4}$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu + \partial_\nu\mathcal{A}_\mu - \mathrm{i}\left(\mathcal{A}_\mu * \mathcal{A}_\nu - \mathcal{A}_\nu * \mathcal{A}_\mu\right), \tag{1.5}$$

and $\mathcal{A}_\mu \equiv \mathcal{A}^a_\mu t^a$ with $\mathrm{tr}(t^a t^b) = \delta^{ab}$. In this article we will only deal with the two-dimensional gauge theories and we have $\theta_{21} = -\theta_{12} = \theta$.

Noncommutative quantum field theories are closely related to the twisted Eguchi-Kawai models (TEK) which have been known [5, 6] since the early 1980's. These models are constructed in such a way [5] that they in the large-N limit reproduce the planar diagrams of corresponding ordinary quantum field theories. This relation was further pursued in refs. [7]–[9] where it was shown how noncommutative quantum field theories can be obtained from the twisted Eguchi-Kawai models in a certain double-scaling limit (see [10] for a review).

In analogy with the twisted reduced models the parameters $\theta_{\mu\nu}$ disappear in planar diagrams of the noncommutative quantum field theories, while for nonplanar diagrams it resides [11] in an additional phase factor of integrands, which is determined by the intersection matrix. As was argued in ref. [12], a nonplanar diagram of genus G is suppressed in $\mathrm{U}_\theta(N)$ noncommutative theories at large θ as

$$\frac{1}{N^{2G}\left(p^{2D}|\det_{\mu\nu}(\theta_{\mu\nu})|\right)^G}, \tag{1.6}$$

where p is a typical value of external momenta. Note that the leading orders both in $1/\theta$ and $1/N$ are governed by the genus of the diagram.

So far, the analysis of this and related noncommutative theories (with matter fields included) have been restricted to a few leading orders of the perturbative expansion in g^2. Not much is known about their nonperturbative quantum dynamics, except for the existence of certain classical solutions [13]–[15].

Before addressing the more complicated problems of nonperturbative quantum dynamics, it is reasonable to begin with a careful examination of the simplest theory — noncommutative (euclidean) gauge theory in two dimensions. The same strategy was followed in the case of ordinary gauge theories and, similarly to the U(N) two-dimensional non-abelian Yang-Mills theory [16], the analysis is greatly simplified by the use of the axial gauge, where the self-interaction of the gauge field disappears. The effects of noncommutativity are still present in the definition of observables, for instance in the averages of noncommutative Wilson loops, $W(C)$, which were introduced in ref. [17] and further examined in refs. [8] and [18]–[25]. In the $N=\infty$ limit of ordinary Yang–Mills theories the Wilson loop averages for contours without self-intersection are given by [26]

$$\langle W(C)\rangle^{(0)}_{\mathrm{U}(N)} = \mathrm{e}^{-\sigma A(C)}\,, \qquad \sigma = \frac{g^2 N}{2}\,, \tag{1.7}$$

where $A(C)$ stands for the area of the surface enclosed by the loop C. Formula (1.7) coincides with the formula obtained for an abelian gauge theory. In contrast to this the Wilson loop averages in the 2D noncommutative gauge theory (1.4) exhibit a nontrivial dependence on θ. References [27]–[37] are devoted to the analysis of 2D noncommutative gauge theory.

In the present paper we analyze 2D euclidean noncommutative gauge theory perturbatively in g^2. We explicitly calculate the contribution of the nonplanar diagram of the order g^4 (having genus 1) to the average of the noncommutative Wilson loop in $\mathbb{R}^2$:

$$\langle W(C)\rangle_{\mathrm{U}_\theta(N)} = \sum_{G=0}^{\infty} N^{-2G}\,\langle W(C)\rangle^{(G)}_{\mathrm{U}_\theta(1)}\,, \tag{1.8}$$

and find that its expansion in $1/\theta$ begins with the term

$$\langle W(C)\rangle^{(1)}_{\mathrm{U}_\theta(1)} = -\frac{\sigma^2}{2\pi^2}\left(1+\frac{\pi^2}{3}\right)A^2(C) + \mathcal{O}(\theta^{-1})\,. \tag{1.9}$$

This anomalous term disagrees with the formula (1.6) and is due to the singular IR behavior of the gauge propagator in two dimensions. As a consequence not only planar diagrams survive as $\theta\to\infty$ in the framework of the perturbative expansion of 2D noncommutative gauge theory.

In the companion paper [38] we evaluate the contribution of *all* diagrams of genus 1 to the noncommutative Wilson loop average (1.8) for a rectangular contour $C=\Box$ and show that at asymptotically large θ it behaves as

$$\langle W(\Box)\rangle^{(1)}_{\mathrm{U}_\theta(1)} \;\longrightarrow\; \frac{4}{\pi^2\,(\sigma\theta)^2}\,\frac{\ln(\sigma A)}{\sigma A} \tag{1.10}$$

for the areas $\sigma^{-1} \ll A(C) \ll \theta$ much larger than the string tension σ introduced in eq. (1.7), but much smaller than θ.[1] In particular, we find that the perturbative and $1/\theta$ expansions of $\langle W(C)\rangle_{\mathrm{U}_\theta(N)}$ are *not* interchangeable and the anomalous terms do not appear within the $1/\theta$-expansion.

[1]In the opposite limit of $\theta/A(C)\to 0$, the Wilson loop averages in two-dimensional noncommutative theory (1.4) approaches the ones for the ordinary U(N) Yang-Mills theory.

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This paper is organized as follows. In section 2, applying the path-integral representation [18] of the star-product, we derive the concise formula (2.7) for a generic Wilson loop average $\langle W(C)\rangle_{\mathrm{U}_\theta(1)}$ in the noncommutative $\mathrm{U}_\theta(1)$ gauge theory (1.4) and discuss how to elevate the abelian results to the generic case of $\mathrm{U}_\theta(N)$. In section 3 we apply the general formula (2.7) in a perturbative calculation of the noncommutative Wilson loop average to order g^4. We demonstrate the appearance of the anomalous term (1.9) for the simplest nonplanar diagram of genus 1 and discuss the associated phenomenon of delocalization. In section 4 we consider the loop equation for the noncommutative Wilson loops in two-dimensional $\mathrm{U}_\theta(1)$ gauge theory and use it to investigate their shape-(in)dependence. In particular, we show that the anomalous term (1.9) is annihilated by the operator on the left-hand side of the loop equation. In section 5 we consider another gauge-invariant observable which is simpler than the noncommutative Wilson loop and which exhibits the same anomalous behavior of the perturbative expansion as $\theta \to \infty$. In section 6 we construct consistent UV/IR regularizations of the two-dimensional noncommutative gauge theory in the axial gauge, using the noncommutative loop equation, and show that the gaussian regularization is compatible with the usual definition of the star-product. Appendix A is devoted to the derivation of the path-integral representation for the noncommutative Wilson loops. Appendix B contains some details of computations for the gaussian regularization. In appendix C we discuss the regularization by a finite box.

2. Generalities

Unless otherwise specified, we will concentrate on the $D = 2$ noncommutative $\mathrm{U}_\theta(1)$ gauge theory — the $N = 1$ option of the $\mathrm{U}_\theta(N)$ noncommutative gauge theory (1.4), defined on the 2D plane $\mathbb{R}^2$. The dependence on N can then be restored using eq. (1.8).

Our aim is to analyze in this theory the average of *closed* Wilson loops[2]

$$W(C) = P\,\mathrm{e}_*^{\mathrm{i}\oint_C \mathrm{d}x_\mu(\tau)\mathcal{A}_\mu(\boldsymbol{x}(\tau))} \tag{2.1}$$

defined via the star-exponential which, as is rederived in appendix A, can be written in the form [18]

$$W(C) = \left\langle \exp\left(\mathrm{i}\oint_C \mathrm{d}x_\mu(\tau)\mathcal{A}_\mu(\boldsymbol{x}(\tau)+\boldsymbol{\xi}(\tau))\right)\right\rangle_\xi\,, \tag{2.2}$$

where the averaging over the auxiliary field $\xi_\mu(\tau)$ is to be performed according to the the path-integral representation

$$\langle \mathcal{B}[\boldsymbol{\xi}(\tau)]\rangle_\xi \stackrel{\mathrm{def}}{=} \int \mathcal{D}\xi_\mu(\tau)\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\theta^{-1})_{\mu\nu}\int \mathrm{d}\tau \mathrm{d}\tau' \xi^\mu(\tau) G^{-1}(\tau,\tau')\xi^\nu(\tau')}\,\mathcal{B}[\boldsymbol{\xi}(\tau)] \tag{2.3}$$

with the standard flat measure.

The smearing function G^{-1} in the exponent in eq. (2.3) has in general support on an interval ε which plays the role of a regularization, and differs, as is discussed in appendix A,

[2]Strictly speaking, the path-ordered exponential (2.1) is invariant under the noncommutative gauge transformations only after the averaging over the gauge fields.

from the naive one which is approached as $\varepsilon \to 0$:

$$G^{-1}(\tau,\tau') \stackrel{\varepsilon\to 0}{\longrightarrow} G_0^{-1}(\tau-\tau') = \dot{\delta}(\tau-\tau')\,, \qquad G_0(\tau-\tau') = \frac{1}{2}\mathrm{sign}\,(\tau-\tau')\,. \tag{2.4}$$

But for the purposes of the present paper, it will be enough to restrict ourselves with the case of $\varepsilon = 0$ which results in the naive form G_0 displayed in eq. (2.4). Then we have

$$\big\langle \xi^\mu(\tau)\,\xi^\nu(\tau')\big\rangle = \frac{\mathrm{i}}{2}\theta^{\mu\nu}\mathrm{sign}\,(\tau-\tau')\,. \tag{2.5}$$

Next, in order to simplify the calculation, let us choose on $\mathbb{R}^2$ the axial gauge

$$\mathcal{A}_1(\boldsymbol{x}) = 0 \implies \mathcal{F}_{\mu\nu} = F_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu\,. \tag{2.6}$$

This gauge choice reduces the $\mathrm{U}_\theta(1)$ action (1.4) (but *not* the average (2.2)) to the one in the ordinary commutative U(1) gauge theory.

We can now interchange the order of averaging over ξ and $\mathcal{A}_2$ — i.e. that over quantum fluctuations of the gauge field — and first calculate the gaussian average over $\mathcal{A}_2$. As a consequence, the ξ-representation (2.2) results in the formula

$$\langle W(C)\rangle_{\mathrm{U}_\theta(1)} = \left\langle \exp\left(-\frac{1}{2}\oint_C \mathrm{d}x_\mu(\tau)\oint_C \mathrm{d}x_\nu(\tau')\,D_{\mu\nu}\left(\boldsymbol{x}(\tau)-\boldsymbol{x}(\tau')+\boldsymbol{\xi}(\tau)-\boldsymbol{\xi}(\tau')\right)\right)\right\rangle_\xi\,, \tag{2.7}$$

where $D_{\mu\nu}(\boldsymbol{z})$ is the standard propagator of the gauge field in $D=2$, which reads in the axial gauge (2.6) as[3]

$$D_{\mu\nu}(\boldsymbol{z}) = \langle\mathcal{A}_\mu(\boldsymbol{z})\mathcal{A}_\nu(\boldsymbol{0})\rangle_{\mathrm{U}(1)} = g^2\,\delta_{\mu 2}\delta_{\nu 2}\left(B-\frac{1}{2}|z_1|\right)\delta^{(1)}(z_2)\,. \tag{2.8}$$

In what follows we shall also need the propagators

$$\langle F_{\mu\nu}(\boldsymbol{z})\mathcal{A}_\lambda(\boldsymbol{0})\rangle_{\mathrm{U}(1)} = -\frac{g^2}{2}\,\epsilon_{\mu\nu}\delta_{\lambda 2}\,\mathrm{sign}\,(z_1)\,\delta^{(1)}(z_2) \tag{2.9}$$

and

$$\langle F_{\mu\nu}(\boldsymbol{z})F_{\rho\lambda}(\boldsymbol{0})\rangle_{\mathrm{U}(1)} = g^2\,\left(\delta_{\mu\rho}\delta_{\nu\lambda}-\delta_{\mu\lambda}\delta_{\nu\rho}\right)\delta^{(2)}(\boldsymbol{z})\,, \tag{2.10}$$

where $\delta^{(2)}(\boldsymbol{z})$ is the standard delta-function in $D=2$, and we used the fact that the propagator in the commutative U(1) theory is given by eq. (2.8).

It is convenient to view the variable τ in eq. (2.7) as an angular variable parameterizing the contour ($\tau\in[0,2\pi]$). The variable $\xi(\tau)$, over which the path integration is to be performed on the right-hand side of eq. (2.7), depends only on the this angular variable τ.

[3]Nothing is expected to depend on the constant B owing to remaining gauge invariance.

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Equation (2.7) contains all information about the Wilson loops on the noncommutative plane. We show below how explicit formulas can be obtained starting from the representation (2.7).

Note that the average (2.7) clearly shows the effects of nonlocality despite the fact that the two-dimensional gauge theory (1.4), irrespectively of the value of θ, has no propagating degrees of freedom (as is manifest in axial gauge).

We also note for future reference that for the general $\mathrm{U}_\theta(N)$ gauge theory the string tension σ entering eq. (1.7), resulting from the contribution of the planar diagrams, is related with g^2 by the formula

$$\sigma = g^2_{\mathrm{U}_\theta(N)} N/2 \,. \tag{2.11}$$

3. Nonplanar diagrams of order g^4

In the present section we compute the first nontrivial g^4-order of the perturbative expansion of the average (2.7) in g^2. For simplicity the contour is assumed to be non-selfintersecting.

3.1 Order g^2

It is instructive first to consider the leading g^2-term. This term is θ-independent and equal to the corresponding contribution in the ordinary commutative $\theta = 0$ abelian U(1) gauge theory. The computation is simple when using the representation (2.7). When expanding the exponential one has to evaluate the appropriate integral of the ξ-average of the propagator

$$\left\langle D_{\mu\nu}(\boldsymbol{x}(\tau) - \boldsymbol{x}(\tau') + \boldsymbol{\xi}(\tau) - \boldsymbol{\xi}(\tau'))\right\rangle_\xi = D_{\mu\nu}(\boldsymbol{x}(\tau) - \boldsymbol{x}(\tau')) \tag{3.1}$$

which, in fact, reduces to the propagator (2.8) itself. To see this introduce the ordinary Fourier representation of the propagator, so that the relevant ξ-average reads

$$\left\langle \mathrm{e}^{\,\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}(\tau)-\boldsymbol{x}(\tau')+\boldsymbol{\xi}(\tau)-\boldsymbol{\xi}(\tau'))}\right\rangle_\xi = \mathrm{e}^{\,\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}(\tau)-\boldsymbol{x}(\tau'))}\cdot \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta_{\mu\nu}p^\mu p^\nu G(\tau,\tau')} = \mathrm{e}^{\,\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}(\tau)-\boldsymbol{x}(\tau'))}\,, \tag{3.2}$$

where the last equality follows from the antisymmetry of $\theta_{\mu\nu}$. Consequently, the g^2-order reproduces the standard abelian result

$$-\frac{g^2}{2}\cdot A(C)\,. \tag{3.3}$$

It is also instructive to use the general formula

$$\langle f(\boldsymbol{\xi}(\tau_1),\boldsymbol{\xi}(\tau_2))\rangle_\xi = \int \frac{\mathrm{d}^D\xi_1^\mu \mathrm{d}^D\xi_2^\mu}{\pi^D|\det\theta|}\,\mathrm{e}^{\,2\mathrm{i}\xi_1^\mu(\theta^{-1})_{\mu\nu}\xi_2^\nu} f(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \tag{3.4}$$

for the ξ-average of a function that depends only on two variables $\xi(\tau_1)$ and $\xi(\tau_2)$ (see appendix A). In our case f depends only on the difference $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$, so that, introducing the variable $\boldsymbol{\eta} = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$, we find

$$\begin{aligned}\int \frac{\mathrm{d}^D\xi_1^\mu \mathrm{d}^D\xi_2^\mu}{\pi^D|\det\theta|}\,\mathrm{e}^{\,2\mathrm{i}\xi_1^\mu(\theta^{-1})_{\mu\nu}\xi_2^\nu} f(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) &= \int \frac{\mathrm{d}^D\xi_1^\mu \mathrm{d}^D\eta^\mu}{\pi^D|\det\theta|}\,\mathrm{e}^{-2\mathrm{i}\xi_1^\mu(\theta^{-1})_{\mu\nu}\eta^\nu} f(\boldsymbol{\eta}) \\ &= \int \mathrm{d}^D\eta^\mu\,\delta^{(D)}(\boldsymbol{\eta}) f(\boldsymbol{\eta}) = f(\boldsymbol{0})\,,\end{aligned} \tag{3.5}$$

which reproduces eq. (3.1).

3.2 The order g^4: planar diagram

Turning to the next-to-leading order g^4, the representation (2.7) leads to the following ξ-average,

$$\langle D_{22}(\boldsymbol{x}(\tau_1) - \boldsymbol{x}(\tau_3) + \boldsymbol{\xi}(\tau_1) - \boldsymbol{\xi}(\tau_3)) \; D_{22}(\boldsymbol{x}(\tau_2) - \boldsymbol{x}(\tau_4) + \boldsymbol{\xi}(\tau_2) - \boldsymbol{\xi}(\tau_4))\rangle_\xi \; . \tag{3.6}$$

or, after the Fourier transformation, to

$$\left\langle \mathrm{e}^{\,\mathrm{i}\boldsymbol{p}_1\cdot(\boldsymbol{\xi}(\tau_1)-\boldsymbol{\xi}(\tau_3))}\, \mathrm{e}^{\,\mathrm{i}\boldsymbol{p}_2\cdot(\boldsymbol{\xi}(\tau_2)-\boldsymbol{\xi}(\tau_4))} \right\rangle_\xi = \mathrm{e}^{\,-\frac{\mathrm{i}}{2}\theta_{\mu\nu} q_k^\mu G_{kj} q_j^\nu} \,, \tag{3.7}$$

where, in compliance with eq. (2.5), we have introduced

$$G_{kj} = \frac{1}{2} \,\mathrm{sign}\,(\tau_k - \tau_j)\,, \tag{3.8}$$

while

$$\boldsymbol{q}_1 = \boldsymbol{p}_1\,, \qquad \boldsymbol{q}_3 = -\boldsymbol{p}_1\,, \qquad \boldsymbol{q}_2 = \boldsymbol{p}_2\,, \qquad \boldsymbol{q}_4 = -\boldsymbol{p}_2\,. \tag{3.9}$$

The ordering of $\boldsymbol{q}_k$ follows the τ_k-ordering of $\boldsymbol{x}(\tau_k)$, i.e. the relative order of the k-labels is the same as the one of the labels τ_k parameterizing the position of the corresponding point $\boldsymbol{x}(\tau_k)$ on the loop C.

The formula (3.7) actually distinguishes the planar diagrams from the remaining non-planar ones. To see this, observe first that the topology of a particular diagram of the perturbation-theory expansion of eq. (2.7) is the same as the one of the corresponding diagram in figure 1. Thus, the planar configurations are selected by the conditions

$$\{\; \tau_1 < \tau_2, \tau_4\,, \quad \tau_3 > \tau_2, \tau_4 \;\}\,, \qquad \{\; 1 \leftrightarrow 2\,, \quad 3 \leftrightarrow 4 \;\}. \tag{3.10}$$

and it follows that the exponent on the right-hand side of eq. (3.7) vanishes

$$\theta_{\mu\nu}\; q_k^\mu \; G_{kj} \; q_j^\nu = 0\,. \tag{3.11}$$

The simplest way to show this is to note that the "clusters" $\{\boldsymbol{q}_1, \boldsymbol{q}_3 = -\boldsymbol{q}_1\}$ and $\{\boldsymbol{q}_2, \boldsymbol{q}_4 = -\boldsymbol{q}_2\}$ do *not* correlate with each other within the combination on the left-hand side of eq. (3.11) as follows from

$$\sum_{j=2,4} G_{1j}\; q_j^\nu = 0\,, \qquad \sum_{j=2,4} G_{3j}\; q_j^\nu = 0\,, \tag{3.12}$$

Figure 1: Diagrams of the order g^4: (a) planar and (b) nonplanar.

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i.e. for the planar diagrams the exponent on the right-hand side of eq. (3.7) reduces to the sum of the ones corresponding to single exchanges (3.2) and vanishes as well. Thus the planar part of the g^4-contribution is θ-independent and has to be equal to the standard $\theta = 0$ result

$$\text{Figure } 1a = \frac{1}{2}\left(\frac{g^2}{2}A(C)\right)^2 . \tag{3.13}$$

Together with the g^2-term (3.3) it provides the first two terms of the expansion of the well-known result $\mathrm{e}^{-g^2 A/2}$ for the loop-average associated to a non-selfintersecting contour in the ordinary commutative U(1) gauge theory.

3.3 Order g^4: nonplanar diagram

The situation is different for the nonplanar diagram of order g^4, depicted in figure 1*b*. To be more specific consider the particular nonplanar assignment

$$\tau_1 < \tau_2 < \tau_3 < \tau_4 . \tag{3.14}$$

A straightforward computation yields

$$\exp\left(-\frac{\mathrm{i}}{2}\theta_{\mu\nu}\, q_k^\mu\, G_{kj}\, q_j^\nu\right) = \exp\left(\mathrm{i}\,\theta_{\mu\nu}\, p_1^\mu\, p_2^\nu\right) . \tag{3.15}$$

Going back to eq. (3.6), one therefore concludes that the average (3.6) reproduces for the nonplanar diagram the *star-product*

$$D_{22}(\boldsymbol{X}) \overset{\bar\theta}{*} D_{22}(\boldsymbol{Y}) = \frac{g^4\theta^2}{4\pi^2}\frac{\exp\left(\mathrm{i}(\theta^{-1})_{\mu\nu}X_\mu Y_\nu\right)}{X_2^2 Y_2^2} \tag{3.16}$$

of the two "propagators" (2.8), where $\boldsymbol{X} = \boldsymbol{x}(\tau_1) - \boldsymbol{x}(\tau_3)$, $\boldsymbol{Y} = \boldsymbol{x}(\tau_2) - \boldsymbol{x}(\tau_4)$ and, generalizing eq. (1.2), we have obtained

$$\begin{aligned} f_1(\boldsymbol{x}) * f_2(\boldsymbol{y}) &= f_1(\boldsymbol{x})\ \exp\left(-\frac{\mathrm{i}}{2}\bar\theta_{\mu\nu}\ \overleftarrow{\partial_\mu^x}\ \overrightarrow{\partial_\nu^y}\right)\ f_2(\boldsymbol{y}) \\ &= \int\prod_{j=1}^{2}\frac{\mathrm{d}^2\xi_j^\mu}{(\pi^2|\det\bar\theta|)^{1/2}}\,\mathrm{e}^{\,2\mathrm{i}(\bar\theta^{-1})_{\mu\nu}\xi_1^\mu\xi_2^\nu} f_1(\boldsymbol{x}+\boldsymbol{\xi}_1) f_2(\boldsymbol{y}+\boldsymbol{\xi}_2) . \end{aligned} \tag{3.17}$$

The parameter of noncommutativity in eq. (3.16) is *twice* larger

$$\bar\theta_{\mu\nu} = 2\theta_{\mu\nu} , \tag{3.18}$$

compared to the original one $\theta_{\mu\nu}$ in eq. (2.3).

Some comments concerning eq. (3.16) are in order. If one applies $-\partial^2/\partial X_1^2$ to eq. (3.16), one obtains

$$\delta^{(2)}(\boldsymbol{X}) \overset{\bar\theta}{*} D_{22}(\boldsymbol{Y}) = \frac{g^2}{4\pi^2}\frac{\exp\left(\mathrm{i}(\theta^{-1})_{\mu\nu}X_\mu Y_\nu\right)}{X_2^2} . \tag{3.19}$$

Acting further by $-\partial^2/\partial Y_1^2$, one reproduces the known formula [12]

$$\delta^{(2)}(\boldsymbol{X}) \overset{\bar\theta}{*} \delta^{(2)}(\boldsymbol{Y}) = \frac{1}{4\pi^2\theta^2}\exp\left(\mathrm{i}(\theta^{-1})_{\mu\nu}X_\mu Y_\nu\right). \tag{3.20}$$

For vanishing X_2 or Y_2, the denominator on the right-hand side of eq. (3.16) (or eq. (3.19)) is to be understood according to the prescription

$$\frac{1}{X_2^2} \to -\frac{\partial}{\partial X_2}\mathcal{P}\left(\frac{1}{X_2}\right), \qquad \frac{1}{Y_2^2} \to -\frac{\partial}{\partial Y_2}\mathcal{P}\left(\frac{1}{Y_2}\right), \tag{3.21}$$

where $\mathcal{P}$ means the principal value.[4]

Equation (3.16) exemplifies a remarkable phenomenon of *long range* dipole-dipole “interactions” (smeared at the scale $\sim \sqrt{\theta}$) between the contour-elements entering into eq. (2.7). The interactions can be traced back to the nonlocality of the star-product (3.17), and is thus built into the noncommutative Wilson loop (2.2).[5] Conceptually this phenomenon of “delocalization” which is enforced by the requirement of the noncommutative gauge invariance for the loop-observables is noteworthy since the 2D noncommutative gauge theory, like ordinary 2D gauge theories, lacks any propagating degrees of freedom. The quasi-locality of the interactions is thus in sharp contrast with the short-range *contact* interactions between the contour elements in the commutative 2D gauge theory with a propagator of the form eq. (2.8).

This phenomenon can be viewed as the generalization of the delocalization emphasized in [12] for the star-product of two δ-functions. In the context of eq. (3.16) the arguments of [12] can be applied for $\boldsymbol{X} = \boldsymbol{Y}$ when the right-hand side of eq. (3.20) becomes constant that, in turn, refers to the infinite range of the “quasi-locality”. The latter infinity precisely matches the $\Delta \to 0$ option of the estimate

$$\delta \sim \max\left(\Delta\,,\ \frac{\theta}{\Delta}\right) \tag{3.24}$$

of the characteristic “width” δ of the star-product (1.2) when the function $f_1(\boldsymbol{x}) = f_2(\boldsymbol{x}) = f(\boldsymbol{x})$ itself has a “width” of order Δ.

[4]This prescription can be justified by calculating the star-product on the left-hand side of eq. (3.16) in a regularized theory along the lines described in section 6, or somewhat simpler in the present context by simply regularizing the propagator according to

$$D_{22}^{(\mathrm{R})}(X) = -\frac{1}{2}|X_1|\,\mathrm{e}^{-\mu|X_1|}\frac{a}{X_2^2+a^2} \qquad \text{or} \qquad D_{22}^{(\mathrm{R})}(p) = \frac{(p_1^2-\mu^2)}{(p_1^2+\mu^2)^2}\,\mathrm{e}^{-a|p_2|}\,. \tag{3.22}$$

The IR regularization, specified by μ, is along the axis 1 and we have simultaneously introduced the UV regularization along the axis 2 as is prescribed by the commutation relation (1.3) which requires $a \sim \theta\mu$. Equation (3.22) has a nice mathematical structure when the regularization vanishes:

$$D_{22}^{(\mathrm{R})}(p) \to -\frac{\partial}{\partial p_1}\mathcal{P}\left(\frac{1}{p_1}\right). \tag{3.23}$$

This reproduces the prescription (3.21), when the regularization is removed.

[5]More complicated nonplanar diagrams of perturbation theory introduce more general multipole interactions between an arbitrary number of “dipoles” made of pairs of the contour elements.

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3.4 Nonplanar order g^4(continued): anomalous behavior

Next, according to the formula (2.7), the right-hand side of eq. (3.16) is to be integrated over those positions of $\boldsymbol{x}(\tau_k)$, $k = 1, \ldots, 4$ on the contour C, which are consistent with the nonplanar topology of the associated diagram. For the nonplanar diagram in figure 1b, we have

$$\text{Figure } 1b \;= g^4 \boldsymbol{P} \int_x \int_y \int_z \int_t D_{22}(X) * D_{22}(Y)\,, \tag{3.25}$$

where explicitly

$$\boldsymbol{P} \int_x \int_y \int_z \int_t \cdots \stackrel{\text{def}}{=} \int_0^{2\pi} \mathrm{d}\tau_x \int_{\tau_x}^{2\pi} \mathrm{d}\tau_y \int_{\tau_y}^{2\pi} \mathrm{d}\tau_z \int_{\tau_z}^{2\pi} \mathrm{d}\tau_t \; \dot{x}_2(\tau_x)\dot{x}_2(\tau_y)\dot{x}_2(\tau_z)\dot{x}_2(\tau_t) \cdots \tag{3.26}$$

with $0 \leq \tau < 2\pi$ parametrizing the loop C, given by the function $x_\mu(\tau)$ $(x_\mu(0) = x_\mu(2\pi))$, and we have introduced

$$\begin{aligned} x &\equiv x(\tau_1)\,, \qquad & y &\equiv x(\tau_2)\,, \\ z &\equiv x(\tau_3)\,, \qquad & t &\equiv x(\tau_4)\,, \end{aligned} \tag{3.27}$$

so that

$$X = x - z\,, \qquad Y = y - t\,. \tag{3.28}$$

Generically, the θ-dependence of the resulting expression does *not* possess any apparent topological interpretation. Moreover, the expansion of the right-hand side of eq. (3.16) in $1/\theta$ starts from the term θ^2 rather than $1/\theta^2$ as one might have expected. However, the θ^2-term and the θ^1-term of the expansion of (3.25) in $1/\theta$ can easily be shown to vanish in accordance with [36].

The θ^0-term of eq. (3.25) reads

$$\theta^0\text{-term} \;= -\frac{g^4}{8\pi^2}\left(\boldsymbol{P}\int_x\int_y\int_z\int_t + \boldsymbol{P}\int_y\int_x\int_t\int_z\right)\left(\frac{X_1^2}{X_2^2} - \frac{X_1 Y_1}{X_2 Y_2}\right). \tag{3.29}$$

The calculation of the θ^0-term given by (3.29) can be performed as follows. We first integrate X_1^2/X_2^2 over y and t to obtain

$$\left(\boldsymbol{P}\int_x\int_y\int_z\int_t + \boldsymbol{P}\int_y\int_x\int_t\int_z\right)\frac{X_1^2}{X_2^2} = -\boldsymbol{P}\int_x\int_z X_1^2 = 2\boldsymbol{P}\int_x\int_z x_1 z_1 = \oint_x x_1 \oint_z z_1 = A^2 \tag{3.30}$$

independently of the form of the contour. For the second term we use the Stokes theorem

$$\oint_C \mathrm{d}x_2\, f(x) = \int_{S(C)} \mathrm{d}\sigma_{12}(x)\partial_1 f(x) \tag{3.31}$$

for the integrals over x and y and obtain

$$\begin{aligned} -2\boldsymbol{P}\int_x\int_y\int_z\int_t \frac{X_1 Y_1}{X_2 Y_2} &= -2\int_S \mathrm{d}\sigma_{12}(x)\int_S \mathrm{d}\sigma_{12}(y)\int_{y_2}^{x_2}\mathrm{d}z_2\int_{z_2}^{x_2}\mathrm{d}t_2\,\frac{1}{(x_2-z_2)(y_2-t_2)} \\ &= -2A^2\left(-\frac{\pi^2}{6}\right) = \frac{\pi^2}{3}A^2, \end{aligned} \tag{3.32}$$

which adds with (3.30) to

$$\theta^0\text{-term} \;= -\frac{g^4}{8\pi^2}\left(1+\frac{\pi^2}{3}\right)A^2\,. \tag{3.33}$$

The θ^{-1}-term in the expansion of eq. (3.16) is

$$\theta^{-1}\text{-term} \;= -\frac{\mathrm{i}g^4}{3!\cdot 4\pi^2\theta}\left(\boldsymbol{P}\int_x\int_y\int_z\int_t - \boldsymbol{P}\int_y\int_x\int_t\int_z\right)\left(\frac{X_1^3 Y_2}{X_2^2} - 3\frac{X_1^2 Y_1}{X_2}\right). \tag{3.34}$$

The explicit calculation of the right-hand side of eq. (3.34) is as follows. The contour integral of the first term vanishes owing to

$$\int_{x_2}^{z_2}\mathrm{d}y_2\int_{z_2}^{x_2}\mathrm{d}t_2\; Y_2 = 0\,. \tag{3.35}$$

The contribution of the second term involves

$$\begin{aligned}
\boldsymbol{P}\int_x\int_y\int_z\int_t \frac{X_1^2 Y_1}{X_2} &= \frac{1}{4}\oint_C \mathrm{d}x_2\int_{C_{xx}}\mathrm{d}z_2\frac{X_1^2}{X_2}\int_{C_{xz}}\mathrm{d}y_2\int_{C_{zx}}\mathrm{d}t_2\;(y_1-t_1)\\
&= \frac{1}{4}\oint_C \mathrm{d}x_2\int_{C_{xx}}\mathrm{d}z_2 X_1^2\left(\int_{C_{xz}}\mathrm{d}y_2\, y_1 + \int_{C_{zx}}\mathrm{d}t_2\, t_1\right)\\
&= -\frac{1}{2}\oint_x x_1\oint_z z_1\oint_y y_1 = -\frac{1}{2}A^3\,.
\end{aligned} \tag{3.36}$$

We thus obtain

$$\theta^{-1}\text{-term} \;= -\frac{\mathrm{i}g^4 A^3}{8\pi^2\theta} \tag{3.37}$$

which is pure imaginary. The sign depends on the orientation of the contour.

The θ^{-2}-term is given by the expression

$$\theta^{-2}\text{-term} = \frac{g^4}{4!\cdot 4\pi^2\theta^2}\left(\boldsymbol{P}\int_x\int_y\int_z\int_t + \boldsymbol{P}\int_y\int_x\int_t\int_z\right)\left(\frac{X_1^4 Y_2^2}{X_2^2} - 4\frac{X_1^3 Y_1 Y_2}{X_2} + 3X_1^2 Y_1^2\right). \tag{3.38}$$

We have not attempted a contour independent calculation of (3.38). However, for a circle, the difference of the two contour integrals in eq. (3.38) can be calculated. The result is

$$R^4\left(\pi^4 + \frac{175\pi^2}{12}\right) = A^4\left(1+\frac{175}{12\pi^2}\right), \tag{3.39}$$

where R is the radius and A is the area. This result (as well as (3.37)) agrees with those of Bassetto et al. [27, 31] for the Wu-Mandelstam-Leibbrandt propagator in Minkowski space. The coefficient in the θ^0-term differs. However, it agrees with the result obtained by the same authors using the principal value presciption for the propagator [27].[6]

However, for a rectangle it is straightforward to perform the integrals in (3.38) and we obtain:

$$A^4\left(\frac{1}{36} + 1 + \frac{3}{2}\right) = \frac{91}{36}A^4\,. \tag{3.40}$$

The conclusion is that the θ^{-2} contribution to the Wilson loop average is *not* shape independent.[7]

[6] We thank A. Bassetto, A. Torrielli and F. Vian for pointing this out to us.

[7] Again we would like to thank A. Bassetto, A. Torrielli and F. Vian for communicating to us that the result (3.39) is invariant under deformations of the circle to an ellipse.

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4. The NC loop equation

The noncommutative loop equation [23]

$$\epsilon^{\mu\nu}\partial_\nu^x \frac{\delta}{\delta\sigma_{12}(x)}\left\langle \mathcal{W}_{\rm clos}(C)\right\rangle_{\mathcal{A}} = \frac{g^2\mathcal{V}}{(2\pi)^2\theta^2}\oint_C \mathrm{d}z^\mu \left\langle \mathcal{W}_{\rm open}(C_{xz})\,\mathcal{W}_{\rm open}(C_{zx})\right\rangle_{\mathcal{A}} \tag{4.1}$$

relates the average of the closed Wilson loop to the correlator of two open Wilson loops given in the axial gauge by

$$\mathcal{W}_{\rm open}(C_{0\eta}) = \int_V \mathrm{d}^2u\, \mathrm{e}_*^{\,\mathrm{i}\int_{C_{u(u+\eta)}}\mathrm{d}x_2 A_2(u+x)}\,\mathrm{e}^{\,\mathrm{i}\eta\wedge u/\theta}\,, \tag{4.2}$$

where $\eta\wedge u \equiv \eta_1 u_2 - \eta_2 u_1$. Following ref. [10], we have introduced in eq. (4.1) a unit volume $\mathcal{V}$ ($\mathcal{V}=1$ for a box). The planar contribution comes from the factorized part of the correlator (the first term on the right-hand side of)

$$\left\langle \mathcal{W}_{\rm open}(C_{xz})\,\mathcal{W}_{\rm open}(C_{zx})\right\rangle = \left\langle \mathcal{W}_{\rm open}(C_{xz})\right\rangle\left\langle \mathcal{W}_{\rm open}(C_{zx})\right\rangle + \left\langle \mathcal{W}_{\rm open}(C_{xz})\,\mathcal{W}_{\rm open}(C_{zx})\right\rangle_{\rm conn} \tag{4.3}$$

which is proportional to the (smeared) δ-function $\delta^{(2)}(x-z)$.

This is because

$$\left\langle \mathcal{W}_{\rm open}(C_{xz})\right\rangle_{A} \propto \int_V \mathrm{d}^2u\,\mathrm{e}^{\,\mathrm{i}(x-z)^\lambda\theta^{-1}_{\lambda\nu}u^\nu} = (2\pi)^2\delta_\mu^{(2)}\left((x-z)\theta^{-1}\right). \tag{4.4}$$

The precise form of $\delta_\mu^{(2)}$ depends on the IR regularization. For a gaussian spherically symmetric IR cutoff we have

$$\int_V \mathrm{d}^2u\cdots = \int \mathrm{d}^2u\,\mathrm{e}^{\,-\mu^2u^2/2}\cdots\,, \qquad V=\frac{2\pi}{\mu^2} \tag{4.5}$$

and

$$\delta_\mu^{(2)}(\eta\theta^{-1}) = \frac{1}{2\pi\mu^2}\,\mathrm{e}^{\,-\eta^2/2\theta^2\mu^2}\,. \tag{4.6}$$

For a box, which possesses only cubic symmetry, we have

$$\delta_\mu^{(2)}(\eta\theta^{-1}) = \prod_{i=1}^{2}\frac{\theta}{\pi\eta_i}\sin\frac{\eta_i}{\theta\mu}\,, \qquad V=\frac{4}{\mu^2}\,. \tag{4.7}$$

The standard δ-function in the factorized term on the right-hand side of the loop equation is reproduced for $\mu\to 0$ as[8]

$$\frac{\mathcal{V}}{(2\pi)^2\theta^2}\left[(2\pi)^2\delta_\mu^{(2)}(\eta\theta^{-1})\right]^2 = V\delta^{(2)}_{\theta\mu/\sqrt{2}}(\eta) \tag{4.8}$$

for the sphere or

$$\frac{1}{(2\pi)^2\theta^2}\left[(2\pi)^2\prod_{i=1}^{2}\frac{\theta}{\pi\eta_i}\sin\frac{\eta_i}{\theta\mu}\right]^2 = V\prod_{i=1}^{2}\frac{\theta\mu}{\pi\eta_i^2}\sin^2\frac{\eta_i}{\theta\mu} \to V\delta^{(2)}(\eta) \tag{4.9}$$

[8] The volume element $\mathcal{V}$ in eq. (4.1) is equal to 1 for the cube and to $2^{d/2}$ for the spherical regularization in d-dimensions.

for the cube. Therefore, the factorized part of the correlator (4.3) reproduces the loop equation of the $N = \infty$ Yang-Mills theory.

Alternatively, the contribution of the connected correlator in eq. (4.3) to the right-hand side of the NC loop equation (4.1) is suppressed at large θ as $1/\theta^2$, so that eq. (4.1) reproduces the loop equation of the $N = \infty$ Yang-Mills theory as $\theta \to \infty$. The expectation that only planar diagrams survive as $\theta \to \infty$ is based, in particular, on this argument.

4.1 Anomalous terms as zero modes

A question which immediately arises is why the existence of the anomalous term (3.33) (or (3.37)) does not contradict the arguments of the previous paragraph. We show in this subsection that the anomalous terms are zero modes of the operator on the left-hand side of eq. (4.1).

Let us first verify how the gauge-invariant loop equation is satisfied in the axial gauge in 2D. To the order g^4 we have for the connected correlator of the two open Wilson loops on the right-hand side:

$$
\begin{aligned}
-\frac{g^4\mathcal{V}}{4\pi^2\theta^2}\oint_C \mathrm{d}z_\nu \int_{C_{xz}} \mathrm{d}y_2 \int_{C_{zx}} \mathrm{d}t_2 \int_V \mathrm{d}^2u \int_V \mathrm{d}^2v D^{(\mathrm{R})}_{22}(u+y-v-t)\,\mathrm{e}^{\,\mathrm{i}(x-z)^\mu\theta^{-1}_{\mu\lambda}(u-v)^\lambda} = \\
= -\frac{g^4}{4\pi^2}V\oint_C \mathrm{d}z_\nu \int_{C_{xz}} \mathrm{d}y_2 \int_{C_{zx}} \mathrm{d}t_2 D^{(\mathrm{R})}_{22}(\theta^{-1}X_2, -\theta^{-1}X_1)\,\mathrm{e}^{\,\mathrm{i}X\wedge Y/\theta}\,, \quad (4.10)
\end{aligned}
$$

where the propagator in the first line is in coordinate space, and that in the second line is in momentum space.

One obtains the same result by acting with the loop operator on the nonplanar diagram in figure 1b as can be seen by applying the area derivative to the ordered exponential:

$$
\frac{\delta}{\delta\sigma_{12}(x)}\left\langle \mathrm{e}_*^{\,\mathrm{i}\int A_2}\right\rangle_{A_2} = \mathrm{i}\left\langle F_{12}(x)\,\mathrm{e}_*^{\,\mathrm{i}\int A_2}\right\rangle_{A_2}, \tag{4.11}
$$

where $F_{12} = \partial_1 A_2$ in the axial gauge.[9] Differentiating eq. (4.11) with respect to x_1, expanding to the order A_2^3 and using eq. (3.16), we obtain to the order g^4

$$
\left\langle \partial_1^x F_{12}(x)\,\mathrm{e}_*^{\,\mathrm{i}\int A_2}\right\rangle_{A_2} = -\frac{g^4}{4\pi^2}\boldsymbol{P}\int_y\int_z\int_t \frac{\mathrm{e}^{\,\mathrm{i}X\wedge Y/\theta}}{X_2^2}, \tag{4.12}
$$

where the ordering goes from x to x along the contour. This reproduces the $\nu = 2$ component of the right-hand side of eq. (4.10). Differentiating eq. (4.11) analogously with respect to x_2:

$$
-\partial_2^x\frac{\delta}{\delta\sigma_{12}(x)}\left\langle \mathrm{e}_*^{\,\mathrm{i}\int A_2}\right\rangle_{A_2} = -\mathrm{i}\left\langle \partial_2^x F_{12}(x)\,\mathrm{e}_*^{\,\mathrm{i}\int A_2}\right\rangle_{A_2} \tag{4.13}
$$

and integrating by parts (the contact terms are mutually canceled), we obtain the $\nu = 1$ component of the right-hand side of eq. (4.10).

[9]As usual when dealing with the loop equation we assume here that the variation of the contour is much smaller than the UV cutoff ($\sim \theta\mu$).

In order to calculate the result of applying the loop operator to the anomalous terms, we note the following. To order θ^0, eq. (4.12) is reduced to

$$\boldsymbol{P}\int_y\int_z\int_t \frac{1}{X_2^2} = -\oint_z = 0 \tag{4.14}$$

which proves that the θ^0-term (3.29) is annihilated by the operator on the left-hand side of the loop equation. The same is true for the θ^{-1}-term (3.34), whose contribution to eq. (4.12) is

$$\mathrm{i}\frac{g^4}{4\pi^2\theta}\boldsymbol{P}\int_y\int_z\int_t\left(\frac{X_1Y_2}{X_2^2}-\frac{Y_1}{X_2}\right). \tag{4.15}$$

The integral of the first term vanishes as in eq. (3.35). Integrating the second term, we find

$$\int_y\int_z y_1 + \int_z\int_t t_1 = \oint_z\oint_y y_1 = 0\cdot A = 0\,. \tag{4.16}$$

The same statements can be made for the θ^0 and θ^{-1} terms of eq. (4.13), given by the $\nu = 1$ component of the right-hand side of eq. (4.10).

We have thus shown that the anomalous θ^0- and θ^{-1}-terms, (3.29) and (3.34), are annihilated by the loop operator and do not contribute to the NC loop equation (4.1).

4.2 Symplectic invariance and shape-independence

The Wilson loop averages in the ordinary 2D Yang–Mills theory depend only on the area enclosed by the loop. It is a consequence of symplectic invariance. If the same holds in the noncommutative case, the area derivative (4.11) should not depend on where the point x is chosen on the contour. Thus symplectic invariance implies

$$\dot{x}_\mu\partial_\mu^x\frac{\delta}{\delta\sigma_{12}(x)}\left\langle \mathrm{e}_*^{\mathrm{i}\int A_2}\right\rangle_{A_2} = \mathrm{i}\left\langle \dot{x}_\mu\partial_\mu^x F_{12}(x)\,\mathrm{e}_*^{\mathrm{i}\int A_2}\right\rangle_{A_2} = 0\,, \tag{4.17}$$

or, using eq. (4.1),

$$\varepsilon_{\mu\nu}\dot{x}_\mu\oint_C \mathrm{d}z^\nu \left\langle \mathcal{W}_{\text{open}}(C_{xz})\,\mathcal{W}_{\text{open}}(C_{zx})\right\rangle_{A_2} = 0\,. \tag{4.18}$$

For non-selfintersecting contours, the factorized part on the right-hand side of eq. (4.3) always obeys eq. (4.18) just as in the commutative case. For the contribution of the nonplanar diagram of order g^4, depicted in figure 1*b*, we have from eq. (4.10)

$$\varepsilon_{\mu\nu}\dot{x}_\mu\oint_C \mathrm{d}z_\nu\int_{C_{xz}}\mathrm{d}y_2\int_{C_{zx}}\mathrm{d}t_2\,\mathrm{e}^{\mathrm{i}X\wedge Y/\theta}\frac{\partial}{\partial X_2}\mathcal{P}\frac{1}{X_2}\,. \tag{4.19}$$

It can be shown that (4.19) vanishes to the orders θ^0 and θ^{-1}, which agrees with what is shown in subsection 3.4. Rather surprisingly, it also vanishes to the order θ^{-2} for the circle. However, it does not vanish for the rectangle as explicit calculations show, in agreement with the conclusion reached in subsection 3.4, namely that the θ^{-2}-terms are not shape independent.

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5. Another observable

The simplest observable for which the discovered anomaly on $\mathbb{R}^2$ shows up is the average of two field strengths located at the points x and $x+\eta$:

$$G(\eta) = \frac{1}{V}\left\langle \int \mathrm{d}^2x\, \mathcal{F}_{12}(x) * \mathrm{e}_*^{\mathrm{i}\int_x^{x+\eta}\mathcal{A}} * \mathcal{F}_{12}(x+\eta) * \mathrm{e}_*^{\mathrm{i}\int_{x+\eta}^{x}\mathcal{A}} \right\rangle_{\mathcal{A}}, \tag{5.1}$$

where the noncommutative phase factors (along certain paths connecting the points x and $x+\eta$) are required by the star-gauge invariance. When these paths are chosen to be straight, the associated contour is dumbbell shaped.

In the axial gauge, where eq. (5.1) takes the form

$$G(\eta) = \frac{1}{V}\left\langle \int \mathrm{d}^2x\, F_{12}(x) * \mathrm{e}_*^{\mathrm{i}\int_x^{x+\eta}A_2} * F_{12}(x+\eta) * \mathrm{e}_*^{\mathrm{i}\int_{x+\eta}^{x}A_2} \right\rangle_{A_2}, \tag{5.2}$$

there are two types of nonplanar diagrams of order g^4 as is depicted in figure 2. They involve: (a) propagators $\langle F_{12}F_{12}\rangle$ and $\langle A_2A_2\rangle$, (b) and (c) two propagators $\langle F_{12}A_2\rangle$, respectively.

The diagram in figure 2a with $\eta = -X$ contributes

$$g^4\frac{\partial^2}{\partial X_1^2}\int_{C_{xz}}\mathrm{d}y_2\int_{C_{zx}}\mathrm{d}t_2\, D_{22}(X) * D_{22}(Y) = -\frac{g^4}{4\pi^2}\frac{1}{X_2^2}\int_{C_{xz}}\mathrm{d}y_2\int_{C_{zx}}\mathrm{d}t_2\,\mathrm{e}^{\mathrm{i}X\wedge Y/\theta}. \tag{5.3}$$

Similarly the diagrams in figure 2b with $\eta = y - x$ gives

$$g^4\frac{\partial^2}{\partial X_1\partial Y_1}\int_{C_{yx}}\mathrm{d}z_2\int_{C_{zx}}\mathrm{d}t_2\, D_{22}(X) * D_{22}(Y) = \frac{g^4}{4\pi^2}\int_{C_{yx}}\mathrm{d}z_2\int_{C_{zx}}\mathrm{d}t_2\,\frac{1}{X_2Y_2}\,\mathrm{e}^{\mathrm{i}X\wedge Y/\theta} \tag{5.4}$$

and the same contribution comes from the diagram in figure 2c with $\eta = t - x$.

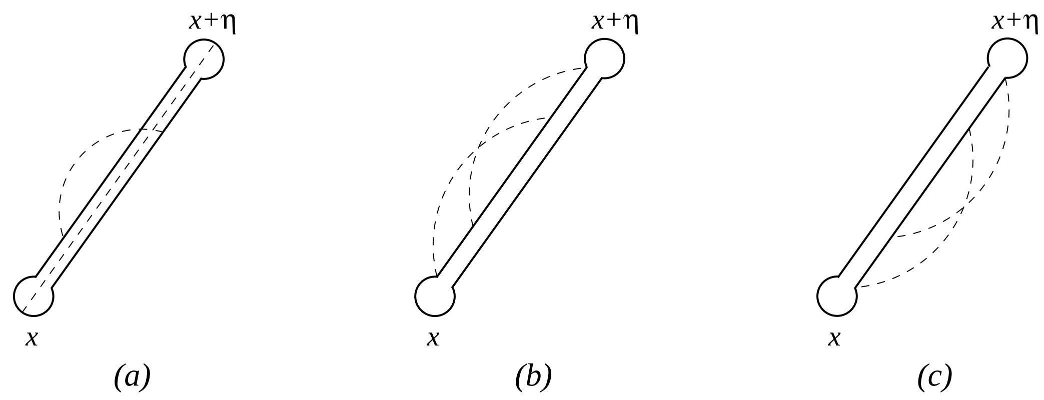

Figure 2: Nonplanar diagrams of the order g^4 for $G(\eta)$ defined by eq. (5.2). The circles at the points x and $x+\eta$ are associated with the field strengths $F_{12}(x)$ and $F_{12}(x+\eta)$, respectively. The noncommutative phase factors along the straight paths connecting them are depicted by the solid lines. The diagrams involve: (a) propagators $\langle F_{12}F_{12}\rangle$ and $\langle A_2A_2\rangle$, (b) and (c) two propagators $\langle F_{12}A_2\rangle$, depicted by the dashed lines.

To the order θ^0 we find from eqs. (5.3) and (5.4)

$$G(\eta) = \frac{g^4}{4\pi^2}\left(1+\frac{\pi^2}{3}\right) \tag{5.5}$$

which does not depend on the form of the paths connecting the points x and $x+\eta$. This formula is in agreement with eq. (3.33) and can be obtained by acting with $-\delta^2/\delta\sigma_{12}(x) \times \delta\sigma_{12}(x+\eta)$ on eq. (3.33), according to eq. (4.11), which is the same as acting with $-\partial^2/\partial A^2$ since the dependence is only on the area.

For a straight path it is convenient to use the variables $Y_2 = y_2 - t_2$ and

$$s_2 = \frac{y_2+t_2}{2} - x_2\,, \qquad s_2 \in [\,0, X_2] \tag{5.6}$$

when integrating over y_2 and t_2 in eq. (5.3) (and similarly in eq. (5.4)). We then find

$$G(\eta) = \frac{g^4}{4\pi^2}\left(1+\frac{\pi^2}{3}\right) \tag{5.7}$$

to all orders in θ^{-1}.

Equation (5.7) is obtained using the "naive" eq. (3.16) which is not applicable, as is already mentioned, for vanishing X_2 or Y_2, when an uncertainty of the type $0\times\infty$ appears in the contour integral. We investigate this issue in appendix B, where some details of the calculation are presented for the gaussian regularization introduced in the next Section. The conclusion is that eq. (5.7) does not change.

6. Consistency of star-product with regularization

As is already mentioned, the "naive" expression (3.16) for the star-product of two propagators is to be regularized. In general, the regularized expression would be regularization-dependent. It is slightly non-trivial to introduce a regularization which preserves what we understand as star-gauge invariance since the star-product mixes the IR and UV sectors of the theory. In this section and appendix B we construct a possible consistent regularization, which is compatible with the NC loop equation and show how the "naive" value given by eq. (3.16) is recovered when the cutoff is removed.

Consistent UV and IR regularizations can be constructed using the noncommutative loop equation (4.1). Introducing an IR cutoff as is described in section 4, we get simultaneously a UV smearing of the delta-function on the right-hand side given by eqs. (4.8) or (4.9) for the spherically or cubic symmetric IR cutoffs, respectively.

The strategy is thus to introduce an IR cutoff by modifying the integral in the definition of the (open) Wilson loop (4.2) that gives eq. (4.4) to the order g^0. As is show in section 4, this fixes the smearing of the δ-function that appears on the right-hand side of the NC loop equation to the order g^2, which in turn fixes the UV regularization of the propagator. This is of course a manifestation of the usual UV/IR mixing in noncommutative theories.

The regularized propagator is then given by the convolution

$$D^{(\mathrm{R})}_{22}(x) = \int \mathrm{d}^2 z\, \delta^{(2)}_{(\mathrm{R})}(z) D^{(0)}_{22}(x-z) \tag{6.1}$$

which obeys

$$-\partial_1^2 D_{22}^{(\mathrm{R})}(x) = \delta_{(\mathrm{R})}^{(2)}(x)\,. \tag{6.2}$$

Given the propagator (6.1), the right-hand side of the loop equation involves to the order g^4 the correlator of two open Wilson loops given by the first line in eq. (4.10) which should be equal to the result of acting by the loop operator on the nonplanar diagram of the order g^4 with two crossed propagator lines. We thus find the following formula is to be valid:

$$V\delta_{(\mathrm{R})}^{(2)}(X) * D_{22}^{(\mathrm{R})}(Y) = \frac{\mathcal{V}}{4\pi^2\theta^2}\int_V \mathrm{d}^2 u \int_V \mathrm{d}^2 v\, D_{22}^{(\mathrm{R})}(u-v+Y)\,\mathrm{e}^{\,\mathrm{i}X\wedge(u-v)/\theta} \tag{6.3}$$

or, applying $-\partial^2/\partial Y_1^2$,

$$V\delta_{(\mathrm{R})}^{(2)}(X) * \delta_{(\mathrm{R})}^{(2)}(Y) = \frac{\mathcal{V}}{4\pi^2\theta^2}\int_V \mathrm{d}^2 u \int_V \mathrm{d}^2 v\, \delta_{(\mathrm{R})}^{(2)}(u-v+Y)\,\mathrm{e}^{\,\mathrm{i}X\wedge(u-v)/\theta} \tag{6.4}$$

as a consequence.

Separating the volume-factor, we thus find the following formula is to be valid for the gaussian regularization:

$$\delta_{(\mathrm{R})}^{(2)}(X) * D_{22}^{(\mathrm{R})}(Y) = \frac{1}{4\pi^2\theta^2}\int \mathrm{d}^2 u\ \mathrm{e}^{\,-\mu^2u^2/4} D_{22}^{(\mathrm{R})}(u-Y)\,\mathrm{e}^{\,\mathrm{i}X\wedge u/\theta} \tag{6.5}$$

or, applying $-\partial^2/\partial Y_1^2$,

$$\delta_{(\mathrm{R})}^{(2)}(X) * \delta_{(\mathrm{R})}^{(2)}(Y) = \frac{1}{4\pi^2\theta^2}\int \mathrm{d}^2 u\ \mathrm{e}^{\,-\mu^2u^2/4} \delta_{(\mathrm{R})}^{(2)}(u-Y)\,\mathrm{e}^{\,\mathrm{i}X\wedge u/\theta} \tag{6.6}$$

as a consequence.

The star-product on the left-hand sides of eqs. (6.5) and (6.6) should be defined in a way for these formulas to be true. This is a consistency of the star-product in the regularized theory with the regularization.

Making the Fourier transformation, it is easy to see that eq. (6.6) is identically satisfied for the gaussian regularization when[10]

$$\delta_{(\mathrm{R})}^{(2)}(X) = \frac{1}{\pi\theta^2\mu^2}\,\mathrm{e}^{\,-X^2/(\theta\mu)^2} \tag{6.7}$$

by the usual definition of the star-product

$$\mathrm{e}^{\,\mathrm{i}pX} * \mathrm{e}^{\,\mathrm{i}qY} = \mathrm{e}^{\,\mathrm{i}pX+\mathrm{i}qY}\,\mathrm{e}^{\,\mathrm{i}p\wedge q\,\theta}\,. \tag{6.8}$$

Similarly, eq. (6.5) is satisfied for an arbitrary function $D_{22}^{(\mathrm{R})}(X)$ because it is linear.

Given this definition of the star-product, we find for the star-product of the regularized propagators:

$$\begin{aligned} D_{22}^{(\mathrm{R})}(X) * D_{22}^{(\mathrm{R})}(Y) = \frac{g^4}{4\pi^2\theta^2}&\int_{-\infty}^{+\infty}\mathrm{d}u\,\mathrm{e}^{\,-\mu^2u^2/4+\mathrm{i}uY_2/\theta}\left(B-\frac{1}{2}|u-X_1|\right)\times \\ &\times\int_{-\infty}^{+\infty}\mathrm{d}v\,\mathrm{e}^{\,-\mu^2v^2/4-\mathrm{i}vX_2/\theta}\left(B-\frac{1}{2}|v-Y_1|\right) \end{aligned} \tag{6.9}$$

which regularizes the "naive" eq. (3.16) for $X_2, Y_2 \lesssim \mu\theta$, reproducing the usual product of the two propagators (2.8) as $\theta \to 0$.

[10]It is explicitly seen from this formula that $\theta\mu$ plays the role of the UV cutoff a as is prescribed by the UV/IR mixing.

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A. Derivation of the representation (2.2)

To derive the path-integral representation (2.2), let us first observe that, applying the integral form (3.17) of the star-product (1.2), we get for the star-product of an *even* number M of functions:

$$f_1(\boldsymbol{x}) * \cdots * f_M(\boldsymbol{x}) = \sum_{\xi_1,\ldots,\xi_M} \mathrm{e}^{\frac{\mathrm{i}}{2}(\theta^{-1})_{\mu\nu}\xi_l^\mu G_{lj}^{-1}\xi_j^\nu} f_1(\boldsymbol{x}+\boldsymbol{\xi}_1)\cdots f_M(\boldsymbol{x}+\boldsymbol{\xi}_M)\,, \tag{A.1}$$

where the measure reads

$$\sum_{\xi_1,\ldots,\xi_M} \cdots = \prod_{j=1}^M \frac{\mathrm{d}^D\xi_j^\mu}{\sqrt{\pi^D|\det\theta|}}\cdots\,, \tag{A.2}$$

while

$$G_{lj}^{-1} = 2(-1)^{l-j+1}\epsilon_{lj}\,, \qquad G_{lj} = -\frac{1}{2}\epsilon_{lj}\,, \tag{A.3}$$

with

$$\epsilon_{lj} = \begin{cases} 1 & l<j \\ 0 & l=j \\ -1 & l>j \end{cases}\,. \tag{A.4}$$

This formula can be proved by induction.

Next, if some (even number) of f's equal 1, the gaussian integral over the proper variables, which the integrand does not depend on, can be performed reproducing eq. (A.1) for the lower number of nontrivial functions. Analogously, we get

$$\begin{aligned} f_1(\boldsymbol{x}) * \ldots * f_M(\boldsymbol{x}) * f_{M+1}(\boldsymbol{x}) &= \sum_{\xi_1,\ldots,\xi_M} \mathrm{e}^{\frac{\mathrm{i}}{2}(\theta^{-1})_{\mu\nu}\xi_l^\mu G_{lj}^{-1}\xi_j^\nu} \times \\ &\quad \times f_1(\boldsymbol{x}+\boldsymbol{\xi}_1)\cdots f_M(\boldsymbol{x}+\boldsymbol{\xi}_M) f_{M+1}\left(\boldsymbol{x}+\sum_{n=1}^M(-1)^n\boldsymbol{\xi}_n\right) \end{aligned} \tag{A.5}$$

for the star-product of an odd number of functions. Therefore, employing the notation (2.3) for the ξ-averaging, eq. (A.1) can be rewritten as

$$\prod_n * f_n(\boldsymbol{x}) = \left\langle \prod_n f_n(\boldsymbol{x}+\boldsymbol{\xi}_n)\right\rangle_\xi\,. \tag{A.6}$$

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In particular, the simplest average is

$$\left\langle \xi_l^\mu \xi_j^\nu \right\rangle_\xi = -\frac{\mathrm{i}}{2}\,\theta^{\mu\nu}\,\epsilon_{lj} \tag{A.7}$$

since the inverse to the matrix G_{ij}^{-1} is given by eq. (A.3).

Finally, let us note the following subtlety. If we make a finite-dimensional approximation of the functional space, say, by means of the stepwise regularization, we formally get from eq. (2.3) the pattern of eq. (2.4). However, the measure in this case is not of the Wiener type, typical trajectories are not continuous[11] and uncertainties of the type $0 \times \infty$ will appear when $\dot{\xi}$ is involved in calculations. We shall rather keep G smeared over an interval $\varepsilon \sim 1/M$ to do the uncertainties, while the results will be independent of the form of the smearing. After doing the uncertainties we set $\epsilon = 0$.

More complicated averages involving a functional $F[\xi]$ can be calculated using the Schwinger-Dyson equation

$$\left\langle \xi^\mu(\tau) F[\xi] \right\rangle_\xi = \mathrm{i}\theta^{\mu\nu} \int \mathrm{d}\tau'\, G(\tau,\tau') \left\langle \frac{\delta F[\xi]}{\delta \xi^\nu(\tau')} \right\rangle_\xi , \tag{A.8}$$

which results from the invariance of $\mathcal{D}\xi$ under an infinitesimal variation of the function $\xi^\mu(\tau)$. In particular, after the substitution $F[\xi] = \xi^\nu(\tau')$, we obtain

$$\left\langle \xi^\mu(\tau)\, \xi^\nu(\tau') \right\rangle_\xi = \mathrm{i}\theta^{\mu\nu} G(\tau,\tau')\,. \tag{A.9}$$

On the other hand, differentiating eq. (A.8) with respect to τ, we obtain another useful formula

$$\left\langle \dot{\xi}^\mu(\tau) F[\xi] \right\rangle_\xi = \mathrm{i}\theta^{\mu\nu} \int \mathrm{d}\tau'\, \dot{G}(\tau,\tau') \left\langle \frac{\delta F[\xi]}{\delta \xi^\nu(\tau')} \right\rangle_\xi \tag{A.10}$$

which is to be employed when $\dot{\xi}$ enters the relevant averages.

Using this technique, eq. (2.2) comes as a result of Feynman's disentangling of the star-products and can be proved by expanding in the powers of $\mathcal{A}$ and using eq. (A.9) whose $\varepsilon \to 0$ limit is given by eq. (2.4). There are no uncertainties at this level since $\dot{\xi}$ is not involved. In particular, this allows us to reduce eq. (A.9) to eq. (2.5).

To illustrate the subtleties with $\dot{\xi}$, let us calculate the variation of $W(C)$ at an intermediate point τ which should reproduce the noncommutative field strength (1.5). Applying the variational derivative to the right-hand side of eq. (2.2), we get[12]

$$\frac{\delta}{\delta x^\mu(\tau)} W(C) = \mathrm{i} \lim_{\varepsilon \to 0} \left\langle \Big(\left(\partial_\mu \mathcal{A}_\nu(\tau) - \partial_\nu \mathcal{A}_\mu(\tau)\right) \dot{x}^\nu(\tau) - \partial_\nu \mathcal{A}_\mu(\tau) \dot{\xi}^\nu(\tau) \Big)\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}x^\rho \mathcal{A}_\rho(\boldsymbol{x}+\boldsymbol{\xi})} \right\rangle_\xi , \tag{A.11}$$

where we denoted $\mathcal{A}_\mu(\tau) \equiv \mathcal{A}_\mu(\boldsymbol{x}(\tau) + \boldsymbol{\xi}(\tau))$ for brevity. Using eq. (A.10) we can replace here $\dot{\xi}^\nu(\tau)$ by

$$\dot{\xi}^\nu(\tau) \overset{\text{w.s.}}{=} -\theta^{\nu\lambda} \int \mathrm{d}\sigma\, \dot{G}(\tau,\sigma) \partial_\lambda \mathcal{A}_\rho(\sigma) \dot{x}^\rho(\sigma)\,, \tag{A.12}$$

[11]This can be directly seen from the form of the matrix G_{ij}^{-1} given by eq. (A.3), which is obviously nonlocal.

[12]An extra term $\mathrm{i}\mathcal{A}_\mu(x(\tau))\left(\delta(\tau-\tau_f) - \delta(\tau-\tau_0)\right)$ emerges at the end points as usual.

which holds in the weak sense, i.e. under the averaging over ξ. This yields explicitly

$$\mathcal{F}_{\mu\nu}(\boldsymbol{x}) = \lim_{\varepsilon\to 0}\Big\langle\Big(\partial_\mu\mathcal{A}_\nu(\boldsymbol{x}) - \partial_\nu\mathcal{A}_\mu(\boldsymbol{x}) + \\ + \theta^{\nu\lambda}\int \mathrm{d}\sigma\,\dot{G}(\tau,\sigma)\partial_\nu\mathcal{A}_\mu(\boldsymbol{x}+\boldsymbol{\xi}(\tau))\partial_\lambda\mathcal{A}_\rho(\boldsymbol{x}+\boldsymbol{\xi}(\sigma))\Big)\Big\rangle_\xi . \quad \text{(A.13)}$$

It is easy to see this is indeed a correct formula expanding in ξ and using eq. (A.9). The combinatorics is as follows

$$\lim_{\varepsilon\to 0}\frac{n!}{n!n!}\int \mathrm{d}\sigma\,\dot{G}(\tau,\sigma)G^n(\tau,\sigma) = \frac{1}{n!}\frac{1}{(n+1)} = \frac{1}{(n+1)!}\,. \quad \text{(A.14)}$$

If we were substitute the limiting value $G_0(\tau-\sigma)$ given by eq. (2.4) when $\dot{G}_0(\tau-\sigma) = \delta(\tau-\sigma)$ into the right-hand side of eq. (A.13) before averaging, we would rather get for the star-product only the term of the first order in θ since $G_0(0)=0$.

A lesson we have learned from this exercise is that whenever $\dot{\xi}(\tau)$ appears inside the averaging it should be substituted according to eq. (A.12) rather than just by its $\varepsilon\to 0$ limit which is given by $-\theta^{\nu\lambda}\partial_\lambda\mathcal{A}_\rho(\tau)\dot{x}^\rho(\tau)$. Stated differently, the star-commutator of two functions is represented in the integrand of the path integral by

$$f(\boldsymbol{x}) * g(\boldsymbol{x}) - g(\boldsymbol{x}) * f(\boldsymbol{x}) \to \mathrm{i}\theta^{\nu\lambda}\int \mathrm{d}\sigma\,\dot{G}(\tau,\sigma)\partial_\nu f(\boldsymbol{x}+\boldsymbol{\xi}(\tau))\partial_\lambda g(\boldsymbol{x}+\boldsymbol{\xi}(\sigma))\,. \quad \text{(A.15)}$$

An application of this formula is to demonstrate the star-gauge covariance of the right-hand side of eq. (2.2) under the star-gauge transformation

$$\delta_\alpha\mathcal{A}_\mu = \partial_\mu\alpha + \mathrm{i}\,(\alpha * \mathcal{A}_\mu - \mathcal{A}_\mu * \alpha)\,. \quad \text{(A.16)}$$

Using eqs. (A.16), (A.15), (A.12), we have

$$\begin{aligned}
\delta_\alpha W(C) &= \lim_{\varepsilon\to 0}\mathrm{i}\Big\langle\int \mathrm{d}\tau\dot{x}^\mu(\tau)\delta_\alpha\mathcal{A}_\mu(\boldsymbol{x}(\tau)+\boldsymbol{\xi}(\tau))\,\mathrm{e}^{\,\mathrm{i}\int \mathrm{d}x^\rho\mathcal{A}_\rho(\boldsymbol{x}+\boldsymbol{\xi})}\Big\rangle_\xi \\
&= \lim_{\varepsilon\to 0}\mathrm{i}\Big\langle\int \mathrm{d}\tau\dot{x}^\mu(\tau)\Big(\partial_\mu\alpha(\tau) - \theta^{\nu\lambda}\int \mathrm{d}\sigma\dot{G}(\tau,\sigma)\partial_\nu\alpha(\tau)\partial_\lambda\mathcal{A}_\mu(\sigma)\Big)\,\mathrm{e}^{\,\mathrm{i}\int \mathrm{d}x^\rho\mathcal{A}_\rho(\boldsymbol{x}+\boldsymbol{\xi})}\Big\rangle_\xi \\
&= \lim_{\varepsilon\to 0}\mathrm{i}\Big\langle\int \mathrm{d}\tau\,\Big(\dot{x}^\mu(\tau)+\dot{\xi}^\mu(\tau)\Big)\,\partial_\mu\alpha(\tau)\,\mathrm{e}^{\,\mathrm{i}\int \mathrm{d}x^\rho\mathcal{A}_\rho(\boldsymbol{x}+\boldsymbol{\xi})}\Big\rangle_\xi \\
&= \lim_{\varepsilon\to 0}\mathrm{i}\Big\langle\int \mathrm{d}\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\,\mathrm{e}^{\,\mathrm{i}\int \mathrm{d}x^\rho\mathcal{A}_\rho(\boldsymbol{x}+\xi)}\Big\rangle_\xi \\
&= \lim_{\varepsilon\to 0}\mathrm{i}\Big\langle\,(\alpha(\tau_f)-\alpha(\tau_0))\,\,\mathrm{e}^{\,\mathrm{i}\int \mathrm{d}x^\rho\mathcal{A}_\rho(\boldsymbol{x}+\boldsymbol{\xi})}\Big\rangle_\xi \\
&= \mathrm{i}\,(\alpha(\boldsymbol{x}(\tau_f)) * W(C) - W(C) * \alpha(\boldsymbol{x}(\tau_0)))
\end{aligned} \quad \text{(A.17)}$$

as it should.

B. Evaluation of integrals for the gaussian regularization

For the gaussian regularization (6.7), we explicitly have from eq. (6.1)[13]

$$D_{22}^{(\mathrm{R})}(X) = g^2\left(B - \frac{1}{2}|X_1|\,\mathrm{Erf}\left(\frac{|X_1|}{\theta\mu}\right) - \frac{\theta\mu}{2\sqrt{\pi}}\,\mathrm{e}^{\,-X_1^2/(\theta\mu)^2}\right)\frac{\mathrm{e}^{\,-X_2^2/(\theta\mu)^2}}{\sqrt{\pi}\theta\mu}\,. \quad \text{(B.1)}$$

[13]Here $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}z\,\mathrm{e}^{\,-z^2}$ is the standard error function.

Differentiating eq. (B.1) with respect to X_1 we find

$$\frac{\partial}{\partial X_1} D_{22}^{(\mathrm{R})}(X) = -\frac{g^2}{2}\mathrm{sign}(X_1)\,\mathrm{Erf}\left(\frac{|X_1|}{\theta\mu}\right)\frac{\mathrm{e}^{-X_2^2/(\theta\mu)^2}}{\sqrt{\pi}\theta\mu}\,. \tag{B.2}$$

Both (B.1) and (B.2) are analytic in X at $X=0$ and reproduce the nonregularized formulas (2.8) and (2.9) for $|X| \gg \theta\mu$.

Given this definition, the star-product on the left-hand side of eq. (6.6) equals

$$\delta^{(2)}_{(\mathrm{R})}(X) * \delta^{(2)}_{(\mathrm{R})}(Y) = \frac{1}{4\pi^2\theta^2}\,\mathrm{e}^{\,\mathrm{i}X\wedge Y/\theta - \mu^2 X^2/4 - \mu^2 Y^2/4}\,. \tag{B.3}$$

It is worth noting that the same expression can be obtained if we do not smear the delta-functions and propagators but rather modify the star-product by including the IR cutoff:

$$\int \frac{\mathrm{d}^2\xi\,\mathrm{d}^2\eta}{4\pi^2\theta^2}\,\mathrm{e}^{\,-\mu^2\xi^2/4-\mu^2\eta^2/4}\,\mathrm{e}^{\,\mathrm{i}\xi\wedge\eta/\theta}\delta^{(2)}(X+\xi)\delta^{(2)}(Y+\eta) = \frac{1}{4\pi^2\theta^2}\,\mathrm{e}^{\,\mathrm{i}X\wedge Y/\theta - \mu^2 X^2/4 - \mu^2 Y^2/4}\,. \tag{B.4}$$

For the star-product of the regularized propagators we analogously obtain eq. (6.9) which can be also rewritten as

$$D_{22}^{(\mathrm{R})}(X) * D_{22}^{(\mathrm{R})}(Y) = \int \frac{\mathrm{d}^2\xi\,\mathrm{d}^2\eta}{4\pi^2\theta^2}\,\mathrm{e}^{\,-\mu^2\xi^2/4-\mu^2\eta^2/4}\,\mathrm{e}^{\,\mathrm{i}\xi\wedge\eta/\theta} D_{22}^{(0)}(X+\xi) D_{22}^{(0)}(Y+\eta) \tag{B.5}$$

in analogy with eq. (B.3).

The right-hand side of eq. (6.9) involves the integrals

$$I(X_1,Y_2) = -\frac{1}{4\pi\theta}\int_{-\infty}^{+\infty} \mathrm{d}u\,\mathrm{e}^{\,-\mu^2u^2/4+\mathrm{i}uY_2/\theta}|u-X_1|\,. \tag{B.6}$$

Differentiating with respect to X_1, as is needed for the observable (5.2) from Sect. 5, we get

$$\begin{aligned}
\frac{\partial I(X_1,Y_2)}{\partial X_1} &= \frac{1}{4\pi\theta}\int_{-\infty}^{+\infty}\mathrm{d}u\,\mathrm{e}^{\,-\mu^2u^2/4+\mathrm{i}uY_2/\theta}\mathrm{sign}(u-X_1) \\
&= \frac{1}{2\pi\theta}\left(-\int_0^{X_1}\mathrm{d}u\,\mathrm{e}^{\,-\mu^2u^2/4}\cos\frac{uY_2}{\theta} + \mathrm{i}\int_{X_1}^{\infty}\mathrm{d}u\,\mathrm{e}^{\,-\mu^2u^2/4}\sin\frac{uY_2}{\theta}\right) \\
&= \frac{1}{2\pi Y_2}\left(-\!\!\!\!\!\!\int_0^{X_1Y_2/\theta}\mathrm{d}\kappa\,\mathrm{e}^{\,-\nu^2\kappa^2/4+\mathrm{i}\kappa} + \mathrm{i}\int_0^{\infty}\mathrm{d}\kappa\,\mathrm{e}^{\,-\nu^2\kappa^2/4}\sin\kappa\right)
\end{aligned} \tag{B.7}$$

where

$$\nu = \frac{\mu\theta}{Y_2}\,. \tag{B.8}$$

As $\nu \to 0$ under normal circumstances, the first integral on the right-hand side yields

$$-\frac{1}{2\pi Y_2}\int_0^{X_1Y_2/\theta}\mathrm{d}\kappa\,\mathrm{e}^{\,\mathrm{i}\kappa} = \mathrm{i}\frac{1}{2\pi Y_2}\left(\mathrm{e}^{\,\mathrm{i}X_1Y_2/\theta}-1\right) \tag{B.9}$$

which itself would not give the anomaly since it vanishes as $\theta \to \infty$. The second integral

$$\mathrm{i}\int_0^{\infty}\mathrm{d}\kappa\,\mathrm{e}^{\,-\nu^2\kappa^2/4}\sin\kappa = \frac{\sqrt{\pi}}{\nu}\,\mathrm{e}^{\,-1/\nu^2}\mathrm{Erf}\left(\frac{\mathrm{i}}{\nu}\right) \to \mathrm{i} \tag{B.10}$$

as $\nu \to 0$. This results in the anomalous behavior of the $1/\theta$-expansion.

To calculate the (regularized) diagrams in figures $2a$ and b, whose contributions are given by the left-hand sides of eqs. (5.3) and (5.4) with the propagators regularized according to eq. (B.1), it is convenient to change the order of integration and first to integrate over the contour and then over ξ and η representing the star-product. This will be also convenient [38] for higher-order calculations.

After the integration over y_2 and z_2, we have for the diagram in figure $2b$

$$\begin{aligned}\text{Figure } 2b = & \frac{g^4}{4\pi^2}\int_0^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta\, \mathrm{e}^{-\tilde{\mu}^2\xi^2/4-\tilde{\mu}^2\eta^2/4} \times \\ & \times \left(\frac{(\cos\xi-1)}{\xi}\frac{(\cos\eta-1)}{\eta} + \frac{\xi^2(\cos\eta-1)-\eta^2(\cos\xi-1)}{\xi\eta(\xi^2-\eta^2)}\right)\end{aligned} \tag{B.11}$$

or

$$\text{Figure } 2b = \frac{g^4}{4\pi^2}\int_0^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta\, \mathrm{e}^{-\tilde{\mu}^2\xi^2/4-\tilde{\mu}^2\eta^2/4}\left(\frac{\cos\xi}{\xi}\frac{\cos\eta}{\eta} + \frac{\eta^2\cos\eta-\xi^2\cos\xi}{\xi\eta(\xi^2-\eta^2)}\right), \tag{B.12}$$

where

$$\tilde{\mu} = \frac{\mu\theta}{R} \tag{B.13}$$

and R is the distance between the points along the axis 2 ($R = |x_2 - y_2|$ for the diagram in figure $2b$). The representations (B.11) and (B.12) are equivalent: the former formula is good for small ξ and η and the latter one is good for large ξ and η, where the integrals are manifestly convergent.

The right-hand side of eq. (B.12) (or eq. (B.11)) can be identically represented for $|x_1 - t_1| \ll 1/\mu$ as

$$\begin{aligned}\text{Figure } 2b = & \frac{g^4}{4\pi^2}\int_0^\infty \frac{\mathrm{d}\xi}{\xi}\int_0^\xi \frac{\mathrm{d}\eta}{\eta}\, \mathrm{e}^{-\tilde{\mu}^2\xi^2/4-\tilde{\mu}^2\eta^2/4}(\cos(\xi-\eta)-\cos\xi) + \\ & + \frac{g^4}{4\pi^2}\int_0^\infty \frac{\mathrm{d}\xi}{\xi}\cos\xi\int_0^\xi \frac{\mathrm{d}\eta}{\eta}\left(\mathrm{e}^{-\tilde{\mu}^2(\xi-\eta)^2/4-\tilde{\mu}^2\eta^2/4} - \mathrm{e}^{-\tilde{\mu}^2\xi^2/4-\tilde{\mu}^2\eta^2/4}\right) + \\ & + \frac{g^4}{2\pi^2}\int_0^\infty \frac{\mathrm{d}\xi}{\xi}\cos\xi\int_0^\xi \frac{\mathrm{d}\eta}{\eta}\left(\mathrm{e}^{-\tilde{\mu}^2\xi^2/2-\tilde{\mu}^2\eta^2/4} - \mathrm{e}^{-\tilde{\mu}^2\xi^2/2+\tilde{\mu}^2\eta^2/4}\right) + \\ & + \frac{g^4}{2\pi^2}\int_0^\infty \frac{\mathrm{d}\xi}{\xi}\cos\xi\int_\xi^\infty \frac{\mathrm{d}\eta}{\eta}\left(\mathrm{e}^{-\tilde{\mu}^2\xi^2/2-\tilde{\mu}^2\eta^2/4} - \mathrm{e}^{-\tilde{\mu}^2\xi^2/4-\tilde{\mu}^2\eta^2/4}\right).\end{aligned} \tag{B.14}$$

The integrals in the second to fourth lines of this equation vanish as $\tilde{\mu} \to 0$, while that in the first line gives

$$\text{Figure } 2b \; = \frac{g^4}{4\pi^2}\frac{\pi^2}{6}\,. \tag{B.15}$$

The contribution of the diagram in figure $2c$ is the same.

After differentiating eq. (6.9) twice with respect to X_1 and integrating over y_2 and t_2, we find that the contribution of the diagram in figure $2a$ for $|X_1| \ll 1/\mu$ is relatively simple:

$$\text{Figure } 2a = \frac{g^4}{2\pi^2}\int_0^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta\, \mathrm{e}^{-\tilde{\mu}^2\xi^2/4-\tilde{\mu}^2\eta^2/4}\delta^{(1)}(\xi)\cos\eta\left(\frac{RB}{\theta} - \frac{1}{2}\eta\right), \tag{B.16}$$

which can be easily integrated as $\tilde{\mu} \to 0$ to give

$$\text{Figure } 2a = \frac{g^4}{4\pi^2}\,. \tag{B.17}$$

Summing the contributions of all three diagrams in figure 2, we get eq. (5.7) which was obtained in section 5 by using the "naive" formula (3.16). We have thus shown that it is reproduced for the gaussian regularization.

C. Regularization by a box

As the star-product of two propagators is regularization-dependent, one may wonder what happens for other regularizations. In this appendix we introduce the regularization by putting the theory in a box and discuss how the "naive" eq. (3.16), which leads us to the nonvanishing θ^0-term, is reproduced. We also speculate how our results can be made compatible with previously obtained results, in particular the results of the TEK model,

Let us introduce an IR regularization by putting the theory in a box of size $L_1 \times L_2$. The star-product in the coordinate space equals

$$\begin{aligned}
D^{(\mathrm{R})}_{22}(X) * D^{(\mathrm{R})}_{22}(Y) &= \int_V \frac{\mathrm{d}^2\xi\, \mathrm{d}^2\eta}{4\pi^2\theta^2}\, \mathrm{e}^{\,\mathrm{i}\xi\wedge\eta/\theta} D_{22}(X-\xi) D_{22}(Y-\eta) \\
&= \frac{g^4}{4\pi^2\theta^2} \int_{-L_1/2}^{+L_1/2} \mathrm{d}\xi_1\, \mathrm{e}^{\,\mathrm{i}\xi_1 Y_2} \left(B - \frac{1}{2}|X_1-\xi_1|\right) \int_{-L_1/2}^{+L_1/2} \mathrm{d}\eta_1\, \mathrm{e}^{\,-\mathrm{i}\eta_1 X_2} \left(B - \frac{1}{2}|Y_1-\eta_1|\right) \\
&= \frac{g^4\theta^2}{4\pi^2} \frac{\left[\mathrm{e}^{\,\mathrm{i}X_1Y_2/\theta} - \left(1 + \mathrm{i}\frac{X_1Y_2}{\theta}\right)\cos\frac{L_1Y_2}{2\theta} + \left(2B - \frac{L_1}{2}\right)\frac{Y_2}{\theta}\sin\frac{L_1Y_2}{2\theta}\right]}{Y_2^2} \times \\
&\quad \times \frac{\left[\mathrm{e}^{\,-\mathrm{i}X_2Y_1/\theta} - \left(1 - \mathrm{i}\frac{X_2Y_1}{\theta}\right)\cos\frac{L_1X_2}{2\theta} + \left(2B - \frac{L_1}{2}\right)\frac{X_2}{\theta}\sin\frac{L_1X_2}{2\theta}\right]}{X_2^2}.
\end{aligned} \tag{C.1}$$

Equation (C.1) differs from (3.16) by terms at most linear in either X_1 or Y_1. These terms vanish when applying $\partial^2/\partial X_1^2$ or $\partial^2/\partial Y_1^2$ and we obtain again eq. (3.20):

$$\delta^{(2)}(X) * \delta^{(2)}(Y) = \frac{1}{4\pi^2\theta^2}\, \mathrm{e}^{\,\mathrm{i}X\wedge Y/\theta}\,. \tag{C.2}$$

If $L_1 \to \infty$ at fixed θ, the extra terms in (C.1) oscillate strongly and can most probably can be omitted, reproducing eq. (3.16). This would be similar to how the one-dimensional propagator

$$D(p) = \int_{-L/2}^{+L/2} \mathrm{d}x \left(B - \frac{1}{2}|x|\right) \mathrm{e}^{\,\mathrm{i}px} = \frac{1}{p^2}\left(1 - \cos\frac{Lp}{2}\right) + \frac{1}{p}\left(2B - \frac{L}{2}\right)\sin\frac{Lp}{2} \tag{C.3}$$

reproduces $1/p^2$ when $Lp \gg 1$, which means that typical distances are much smaller than L. Note that nothing depends on the constant B for such distances, while the formulas of this Section are simplified for $B = L_1/4$.

Alternatively, if $\theta \to \infty$ at fixed L_1 (like on a torus), $L_1/\theta \to 0$ and we obtain from eq. (C.1)

$$D_{22}(X) * D_{22}(Y) = \frac{g^4\theta^2}{4\pi^2} \frac{\left(\mathrm{e}^{\,\mathrm{i}X_1Y_2/\theta} - 1 - \mathrm{i}\frac{X_1Y_2}{\theta}\right)}{Y_2^2} \frac{\left(\mathrm{e}^{\,-\mathrm{i}X_2Y_1/\theta} - 1 + \mathrm{i}\frac{X_2Y_1}{\theta}\right)}{X_2^2}, \tag{C.4}$$

and the expansion in $1/\theta$ begins with a term $\propto \theta^{-2}$. Therefore, the terms of the order θ^0 and θ^{-1} would vanish.

On a torus with periods L_1 and L_2 we have

$$\theta = \frac{L_1 L_2}{2\pi}\Theta\,, \tag{C.5}$$

where (the irrational) Θ is the dimensionless noncommutativity parameter. Using eq. (C.5) we find

$$\frac{L_1X_2}{2\theta} = \frac{\pi}{\Theta}\frac{X_2}{L_2} \tag{C.6}$$

which is small for $\Theta \gtrsim 1$ and $X_2 \ll L_2$, i.e. for the case of loops much smaller than the period, as is required for an approximation of $\mathbb{R}^2$ by $\mathbb{T}^2$. Then eq. (C.4) is reproduced. One might also expect than $L_1X_2/2\theta$ is a multiple of 2π for the regularization by a *discrete* torus, so then eq. (C.4) is exact, thereby explaining the relation with genus expansion in TEK.

Remarkably, eq. (C.4) (or even more general eq. (C.1)) is consistent with the NC loop equation. Applying the loop operator to the nonplanar diagram of the order g^4 involving the star-product (C.1), we get for the IR regularization by a box:

$$\begin{aligned}
&-\frac{g^4}{4\pi^2}V\oint_C \mathrm{d}z_\nu \int_{C_{xz}} \mathrm{d}y_2\,\times \qquad\qquad (\text{C.7})\\
&\times\int_{C_{zx}} \mathrm{d}t_2 \frac{\mathrm{e}^{\,\mathrm{i}X_1Y_2/\theta}\left[\mathrm{e}^{\,-\mathrm{i}X_2Y_1/\theta} - \left(1 - \mathrm{i}\frac{X_2Y_1}{\theta}\right)\cos\frac{L_1X_2}{2\theta} + \left(2B - \frac{L_1}{2}\right)\frac{X_2}{\theta}\sin\frac{L_1X_2}{2\theta}\right]}{X_2^2}\,.
\end{aligned}$$

For the right-hand side of the loop equation we use eq. (4.10), which has in the first line the same integrals as in eq. (C.1). This results in the same expression as (C.7), which replaces the second line of (4.10).

It is worth noting that the modification (C.4) does not cure the shape-dependence of the θ^{-2}-term. It now equals

$$\begin{aligned}
\theta^{-2}\text{-term} &= \frac{g^4}{96\pi^2\theta^2}\left(1 + \frac{5}{\pi^2}\right)A^4 \qquad \text{for circle}\\
\theta^{-2}\text{-term} &= \frac{g^4}{96\pi^2\theta^2}\cdot\frac{3}{2}A^4 \qquad \text{for rectangle}\,,
\end{aligned} \tag{C.8}$$

and we still seem to have an explicit breaking of symplectic invariance in the $1/\theta$-expansion.

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